Problem solving strategies

1. CONCEPTUAL MISTAKES

Questions in competitions are posed to test the students in multiple ways. In general they test them conceptually and strategically, in the sense that examiner will tempt the student to commit mistakes if he/she is not alert.

Let us look at one such example.

$$\int_{-1}^{3} \text{Tan}^{-1}\left(\frac{x}{x^2+1}\right)+\text{Tan}^{-1}\left(\frac{x^2+1}{x}\right)dx \,.$$

By seeing problem student is tempted to write the solution as follows

$$\int_{-1}^{3} \text{Tan}^{-1}\left(\frac{x}{x^2+1}\right)+Cot^{-1}\left(\frac{x}{x^2+1}\right)dx$$

$$= \int_{-1}^{3} \frac{\pi}{2}dx = 4\times\frac{\pi}{2} = 2\pi$$

But in fact answer is wrong.

If you are alert then you will come to know where you made the mistake.

$$\text{Tan}^{-1}(x) = \begin{cases} Cot^{-1}\left(\dfrac{1}{x}\right) \; for & x>0 \\ -\pi + Cot^{-1}\left(\dfrac{1}{x}\right) \; for & x<0 \end{cases}$$

$$\frac{x}{x^2+1} < 0 \; if \;\; x<0, \; \frac{x}{x^2+1} > 0 \; if \;\; x>0$$

$$\text{Tan}^{-1}\left(\frac{x}{x^2+1}\right) = -\pi + Cot^{-1}\left(\frac{x^2+1}{x}\right) \text{ if } -1 < x < 0$$

$$\text{Tan}^{-1}\left(\frac{x}{x^2+1}\right) = Cot^{-1}\left(\frac{x^2+1}{x}\right) \text{ if } 1 < x < 3$$

$$\int_{-1}^{3} \text{Tan}^{-1}\left(\frac{x}{x^2+1}\right)+\text{Tan}^{-1}\left(\frac{x^2+1}{x}\right)dx =$$

$$= \int_{-1}^{0}\left[\text{Tan}^{-1}\left(\frac{x}{x^2+1}\right)+C\text{ot}^{-1}\left(\frac{x}{x^2+1}\right)-\pi\right]dx + \int_{0}^{3}\text{Tan}^{-1}\left(\frac{x}{x^2+1}\right)+C\text{ot}^{-1}\left(\frac{x}{x^2+1}\right)dx$$

$$= \int_{-1}^{0}\left(\frac{\pi}{2}-\pi\right)dx + \int_{0}^{3}\frac{\pi}{2}dx = \pi$$

2. STANDARD QUESTIONS - EASY APPROACHES

There are many questions of the same format or nearer to the same model they can be solved with the specific formulae approach.

$\int \frac{x^2+x+1}{(x-1)(x-2)(x-3)}dx = A\log|x-1|+B\log|x-2|+C\log|x-3|$ where value of A can be obtained by substituting x =1 in the remaining part of the rational function other than (x -1), i.e $A=\frac{1+1+1}{(1-2)(1-3)}=\frac{3}{2}$, Similarly we obtain B = $\frac{7}{-1}=-7$ and C = $\frac{13}{2}$ values.

3. GRAPHICAL APPROACH

Some problems cannot be solved by direct methods. Few of them can be solved by graphical approach.

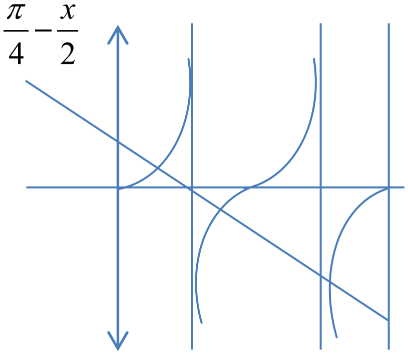
The number of roots of the equation $x+2\tan x=\frac{\pi}{2}$ in the interval $[0,2\pi]$.

We cannot solve directly since it is a combination of trigonometric and algebraic functions.
Graphical approach is helpful to
Decide the number of solutions.
Problem can be re written as

$\tan x=\frac{\pi}{4}-\frac{x}{2}$

Through graph we can observe that there
Are only 3 point of intersections

$y=\frac{\pi}{4}-\frac{x}{2}$



4. SUBSTITUION& ELIMINATION

Some typical generalized problems can weaved just by substitution. Multiple choice advantages can be taken in perceiving the correct answer by end after eliminating remaining options in solving some hard problems.

Problem: Out of $(2n + 1)$ tickets consecutively numbered three are selected at random. Then the probability that they are in A.P

1) $\frac{3n}{4n^2-1}$ 2) $\frac{6n}{4n^2-1}$ 3) $\frac{4n}{4n^2-1}$ 4) $\frac{5n}{4n^2-1}$

Sol: Number of 3 consecutively numbered tickets with common Difference '1' among 2n +1 numbers are 2n -1

Similarly common difference 2, 3, 5,........2n- 3, 2n-5,.....1

Sum of all the numbers = 1 + 3 +5 + ...2n-1 = n^2

Any three numbers can be selected from '2n+1' numbers can be selected in ${}^{2n+1}C_3$ Ways.

Required Probability = $\frac{n^2}{{}^{(2n+1)}C_3} = \frac{3n}{4n^2-1}$

We can get the answer for the same problem by just taking n = 1

$\Rightarrow$ **2n + 1=3.**

There will be only three consecutive numbers {1, 2, 3 }.Three numbers from 3 numbers can be selected in 1 way. Hence probability = 1.

Substituting n = 1 in the options we get 1, 2, 4/3, 5/3 respectively. Obviously first option is correct.

5. TRICKY QUESTIONS

Problem seems to be lengthy, but can be solved with in no time by using trick

The number of solutions of equation $8\left[x^2-x\right]+4\left[x\right]=13+12\left[\sin x\right]$ is (here [.] Represents greatest integer less than or equal to 'x')

Seems to be difficult, we may feel that we need case wise analysis required but careful observation says that [ ] is an integer and terms of LHS are multiples of even numbers. Hence L.H.S is even. Whereas R.H.S is odd since sum of even and odd is always odd number. Hence LHS and RHS never equal.

Therefore number of solutions is equal to zero.

6. STRATEGY INVOLVING QUESTIONS.

In spite of knowing concepts and formulae, we may not be able to solve problems immediately; one should read the problem thoroughly and try to solve it by using operations, definitions and the properties to overcome the hidden logic. Let us look at one such problem

Problem: For any integer and two $n \times n$ matrices with real entries A, B that satisfy the equation $A^{-1} + B^{-1} = (A+B)^{-1}$, detA = 2 then find detB =

Sol: There is no formula to expand $(A+B)^{-1}$. We don't know about the entries of the matrix, and then what is the way to get the solution for the given question.

Shall we use properties of matrix to solve it? Can I take inverse both sides, yes I can, but unfortunately we don't have property on $(A+B)^{-1}$ like $(A+B)^{T}$.

Let us multiply both sides by (A + B)

$(A+B)^{-1}(A+B) = \left(A^{-1} + B^{-1}\right)\left(A+B\right)$

$\Rightarrow I = A^{-1}A + A^{-1}B + B^{-1}A + B^{-1}B$

$\Rightarrow I = I + A^{-1}B + B^{-1}A + I$

$\Rightarrow A^{-1}B + B^{-1}A + I = 0$

We got one relation $A^{-1}B + B^{-1}A + I = 0$, still we are not able to find detB

Let $A^{-1}B = X \Rightarrow B = XA$

$\left(A^{-1}B\right)^{-1} = X^{-1} \Rightarrow B^{-1}A = X^{-1}$

$\Rightarrow A^{-1}B + B^{-1}A + I = 0$ Is converted in the form $X + X^{-1} + I = 0$.We cannot solve this problem like solving algebraic equation.

Multiplying both sides with X(X – I) we get $X^{3} - I = 0$.

$\Rightarrow X^{3} = I \Rightarrow \det\left(X^{3}\right) = \left(\det X\right)^{3} = \det I \Rightarrow \det X = 1$

$\Rightarrow$detB = det(XA) = detX.detA = detA = 2